The cost of factoring and “post-quantum RSA”

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Outline

- Brief history of factoring and of estimating the difficulty of factoring.
- Quantum computing crash course and Shor's algorithm.
- Quantum computing cost metrics and how trying (and failing) to break "post-quantum RSA" changed how I think about expensive quantum algorithms.

Factoring integers

Fundamental theorem of arithmetic:

Every integer n > 1 can be written uniquely as a product of prime powers

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

where $p_1 < p_2 < \ldots < p_k$ are distinct primes and the e_i are positive integers .

Definitions:

To ***factor*** n is to write n as above.

To ***split*** n is to find some integer d, 1 < d < n, that divides n.

How large of a number will we ever factor?

From Odlyzko, “Future of factoring.” 1995.

- 1874, Jevons conjectures nobody (but himself) would ever know the factors of 8,616,460,799.
- 1967, Brillhart and Selfridge: “nothing but frustration can be expected from an attack on a number of 25 or more digits”.
- 1976, Guy: “I shall be surprised if anyone regularly factors numbers of size 10^80 without special form in the present century.”
- 1977, Rivest: factoring 129 digit number would take “40 quadrillion years”.

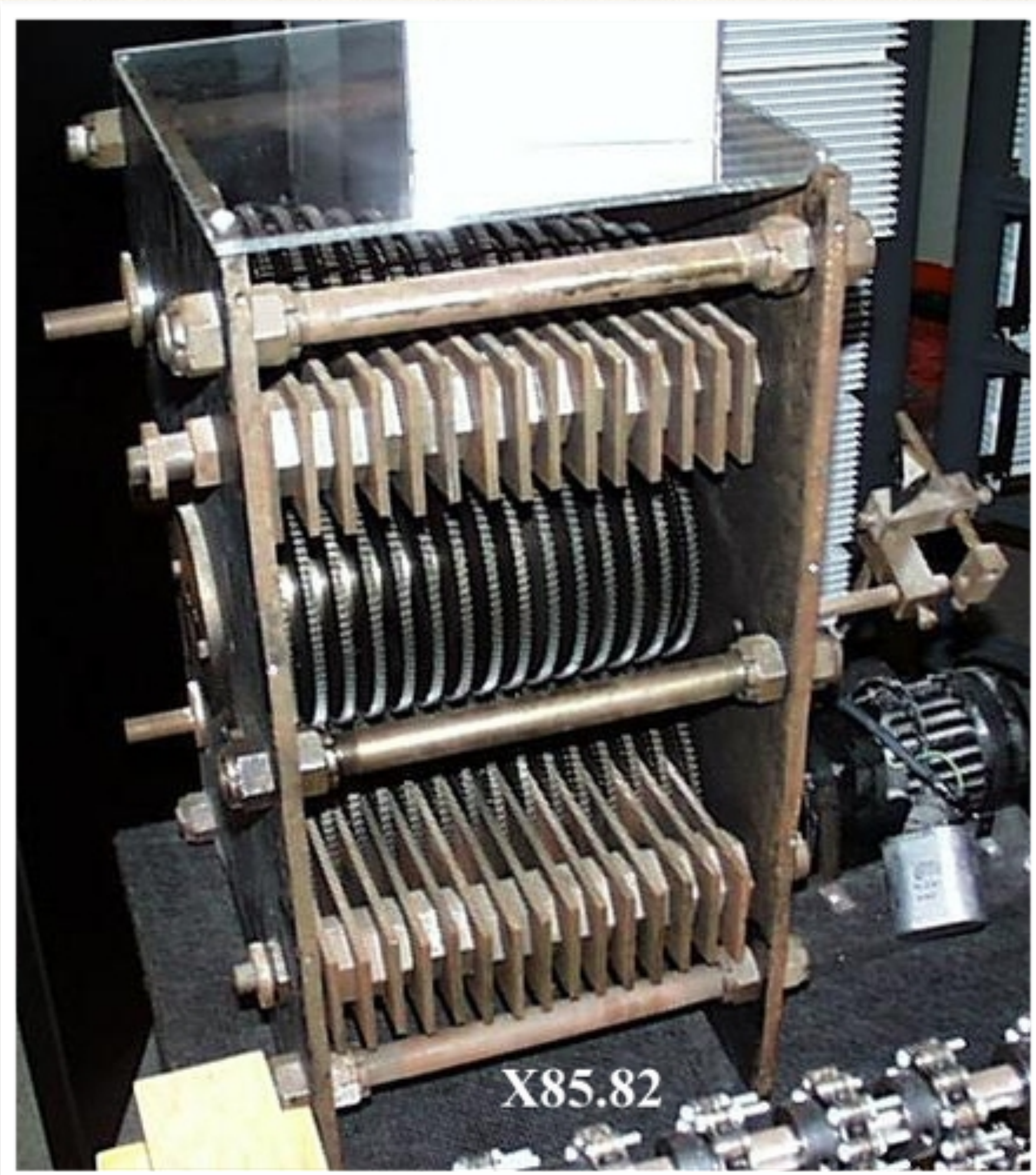
1931: Lehmer's photoelectric gear sieve

Fermat's method

$n = uv$ $\qquad x = (u + v)/2$ $\qquad y = (u - v)/2$

$n = (x + y)(x - y)$ $\qquad x^2 - n = y^2$

- Pick t (small) integers, m1, m2, ..., mt.
- Make t tables. Table i contains quadratic residues r in Z/mi for which r+n is also a quadratic residue.
- Iterate over candidates C for x^2 - n.
- Check if C mod mi is in table i for all i=1...t.
- If C passes all t tests, try to factor n.



1931: Lehmer–Powers describe CFRAC
1970: Morrison–Brillhart implement CFRAC

- 39 digit "F7" factored in September 1970 on an IBM 360/91. Not reported until 1975 Lehmer tribute issue of *Math. Comp*.
- "In those days, integer factorization was not fashionable" (Odlyzko 1995).
- 45 digit factorizations may have been possible.

$$F_7 = 2^{2^7} + 1 = 340{,}282{,}366{,}920{,}938{,}463{,}463{,}374{,}607{,}431{,}768{,}211{,}457$$

1977: Rivest–Shamir–Adleman Cryptosystem

Public key : $n = pq$

Private key : d such that $3d \equiv 1 \pmod{\varphi(n)}$

Encryption

$$\mathscr{E}_n(m) := m^3 \bmod n$$

Decryption

$$\mathscr{D}_n(c) := c^d \bmod n$$

Notes:

- p and q should be primes of roughly equal size that are both congruent to 2 mod 3.
- $\varphi(n)$ is Euler's totient function.

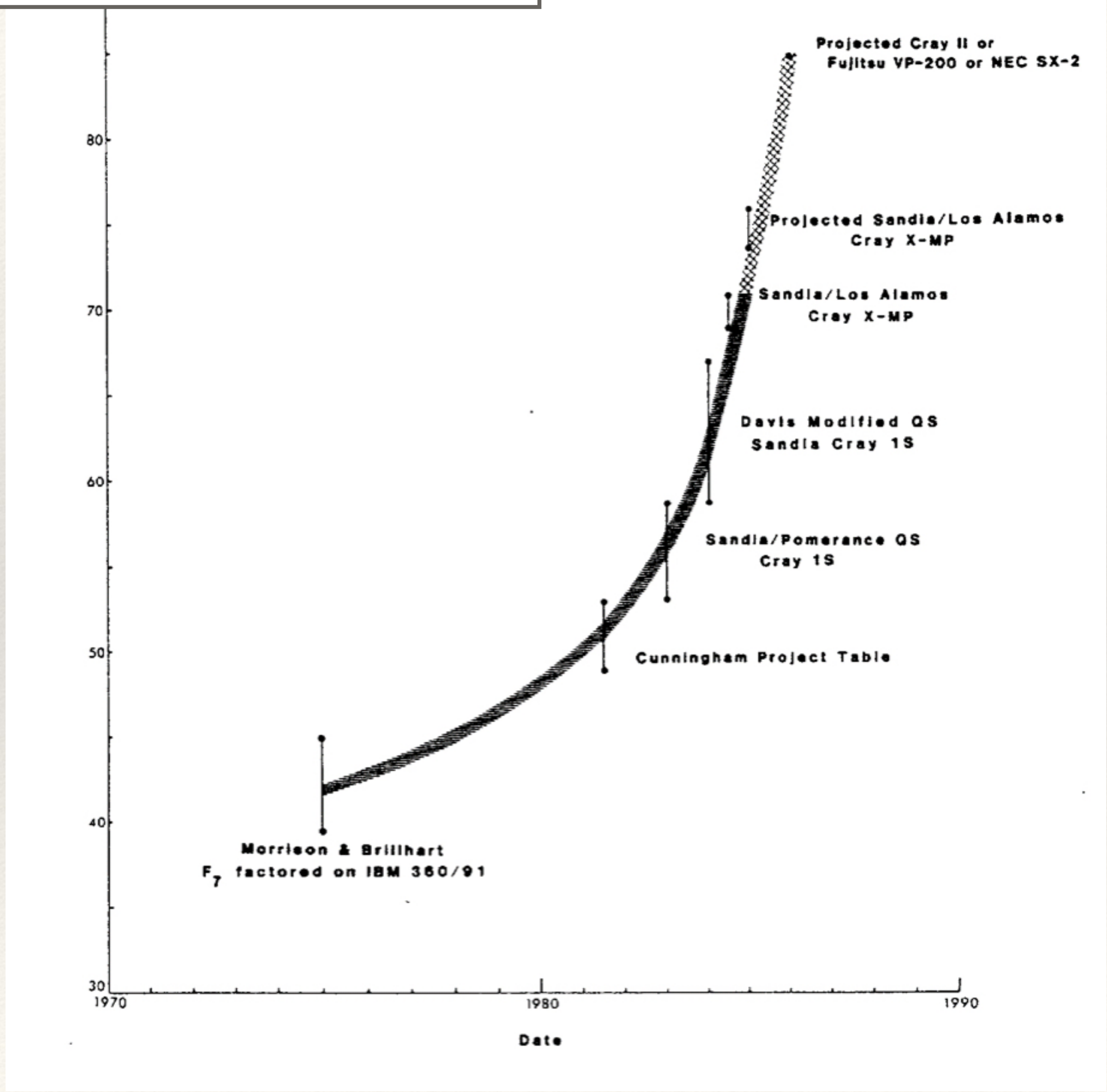
$$\varphi(n) = |(\mathbb{Z}/n)^*| \qquad \varphi(pq) = (p-1)(q-1)$$

1981: Pomerance describes quadratic sieve
1982: Davis–Holdridge–Simmons implement quadratic sieve

- Factors numbers of roughly twice the bitlength that CFRAC can handle.
- Easier to parallelize than CFRAC.
- 1983: Canfield—Erdős—Pomerance give a lower bound on the number of "smooth" numbers in an interval [1,x]. Pomerance conjectures quadratic sieve has complexity

$$\exp\left(\sqrt{\log n \log\log n}\right).$$

“The easiest question to ask concerning integer factoring and the hardest to answer, is; ‘How large a number is it computationally feasible to factor using a general purpose factoring routine?’”(Davis—Holdridge—Simmons, 1984)



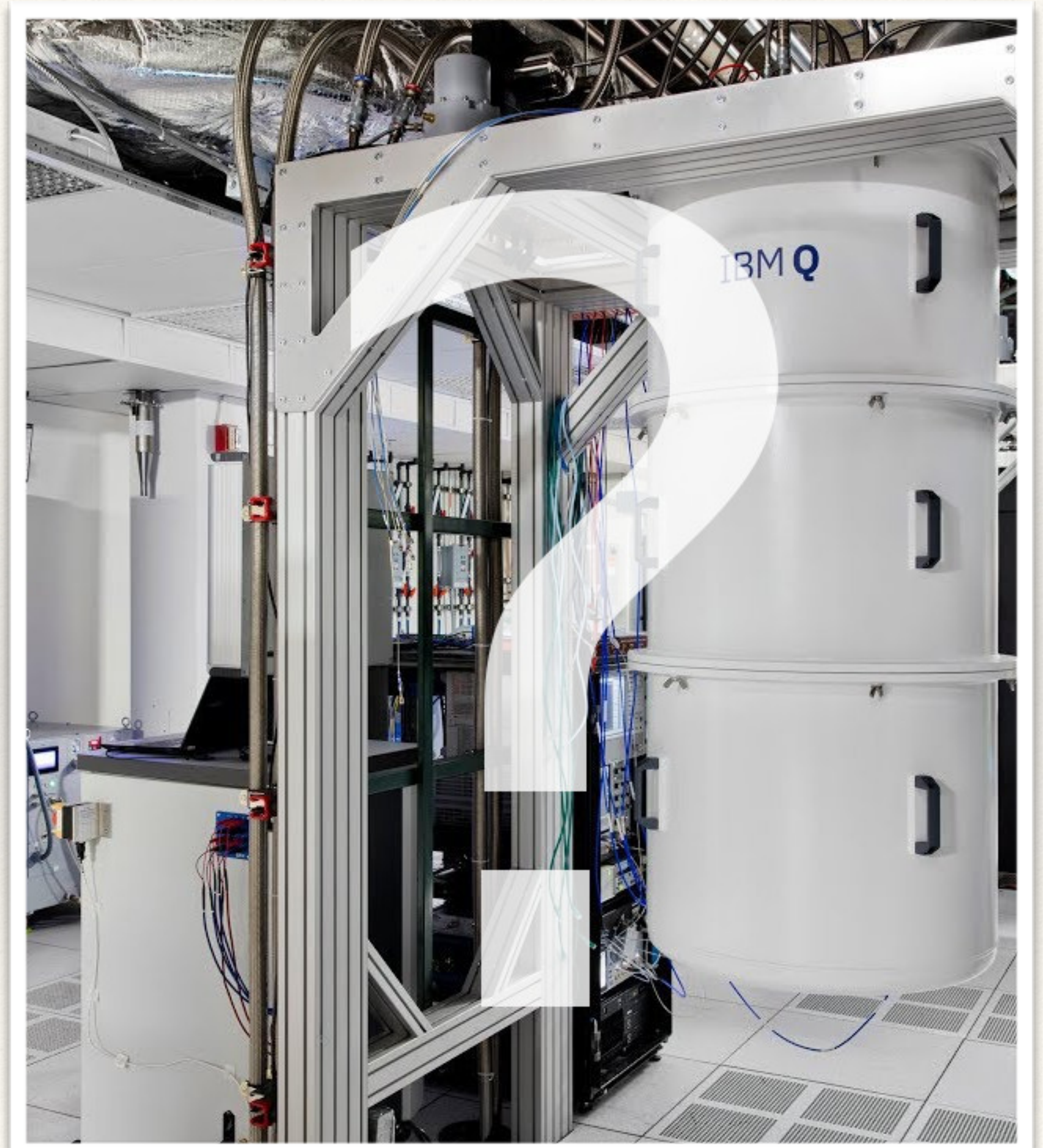
How large of a number will we ever factor?

More recently:

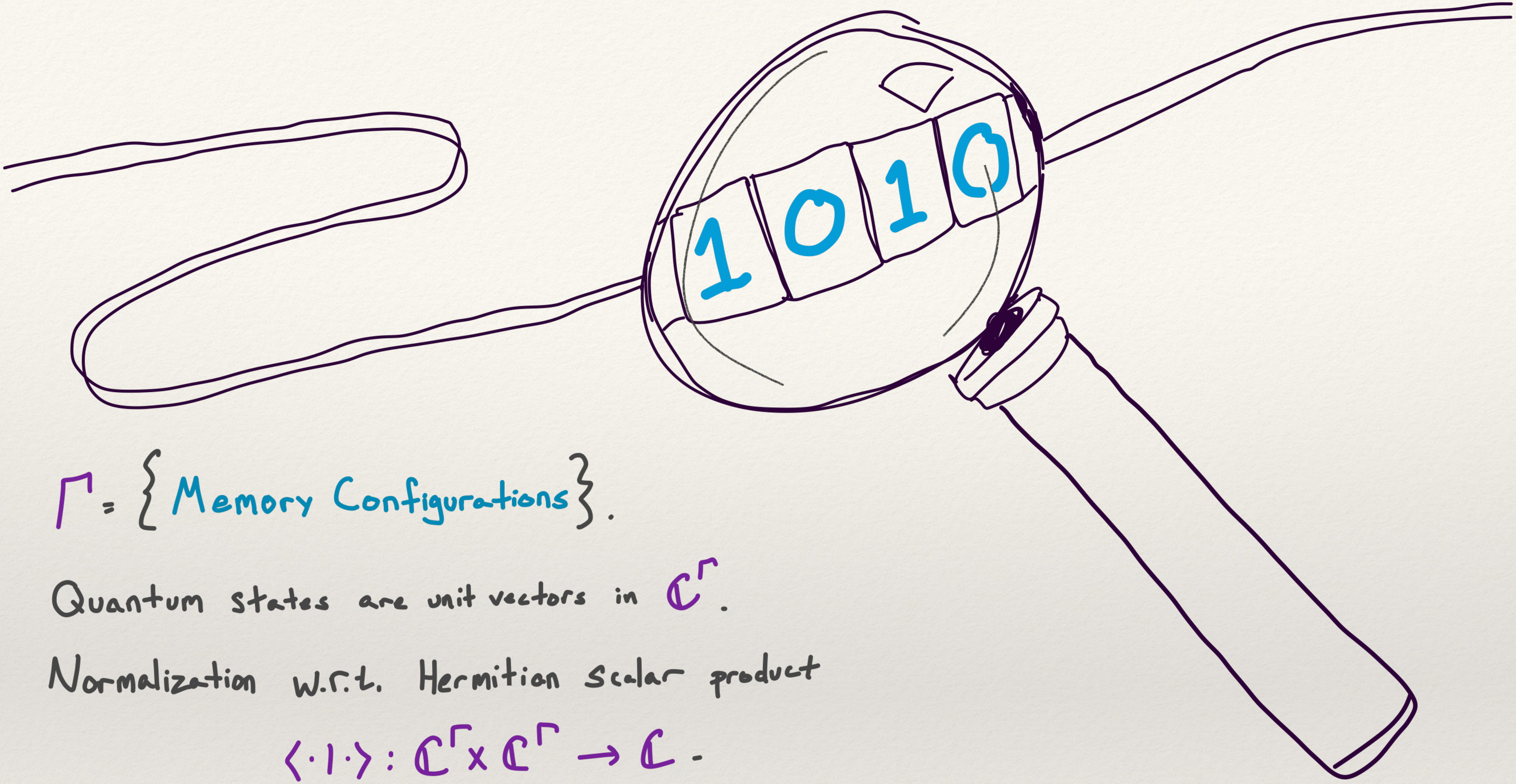
- 2009, Kleinjung—Aoki—Franke—Lensta—Thomé—Bos—Gaudry—Krupa—Montgomery—Osvik—Riele—Timofeev—Zimmerman: Report factorization of 768 bit (232 digit) number. “preciously little doubt about the feasibility by the year 2020” of 1024 bit factorization.
- 2015, NSA: 3072 bit (924 digit) numbers will be hard to factor in short term.
- 2017, Bernstein—Heninger—Lou—Valenta: 2^43 bit (i.e. terabyte) numbers will be hard to factor in long term.

1994: Shor describes quantum factoring algorithm (Yet to be implemented)

- Uses a randomized reduction to order finding.
- Randomly select $a \in (Z/n)^{\times}$.
- Compute order of a, i.e. least r such that $a^r \equiv 1 \pmod{n}$
- (For odd composite n more than half of the elements of (Z/n)* have even order.)
- Check $\gcd(a^{r/2} - 1, n)$
- Splits n as long as r is even and $a^{r/2} \not\equiv -1 \pmod{n}$



Quantum computing



$\Gamma = \{\text{Memory Configurations}\}$.

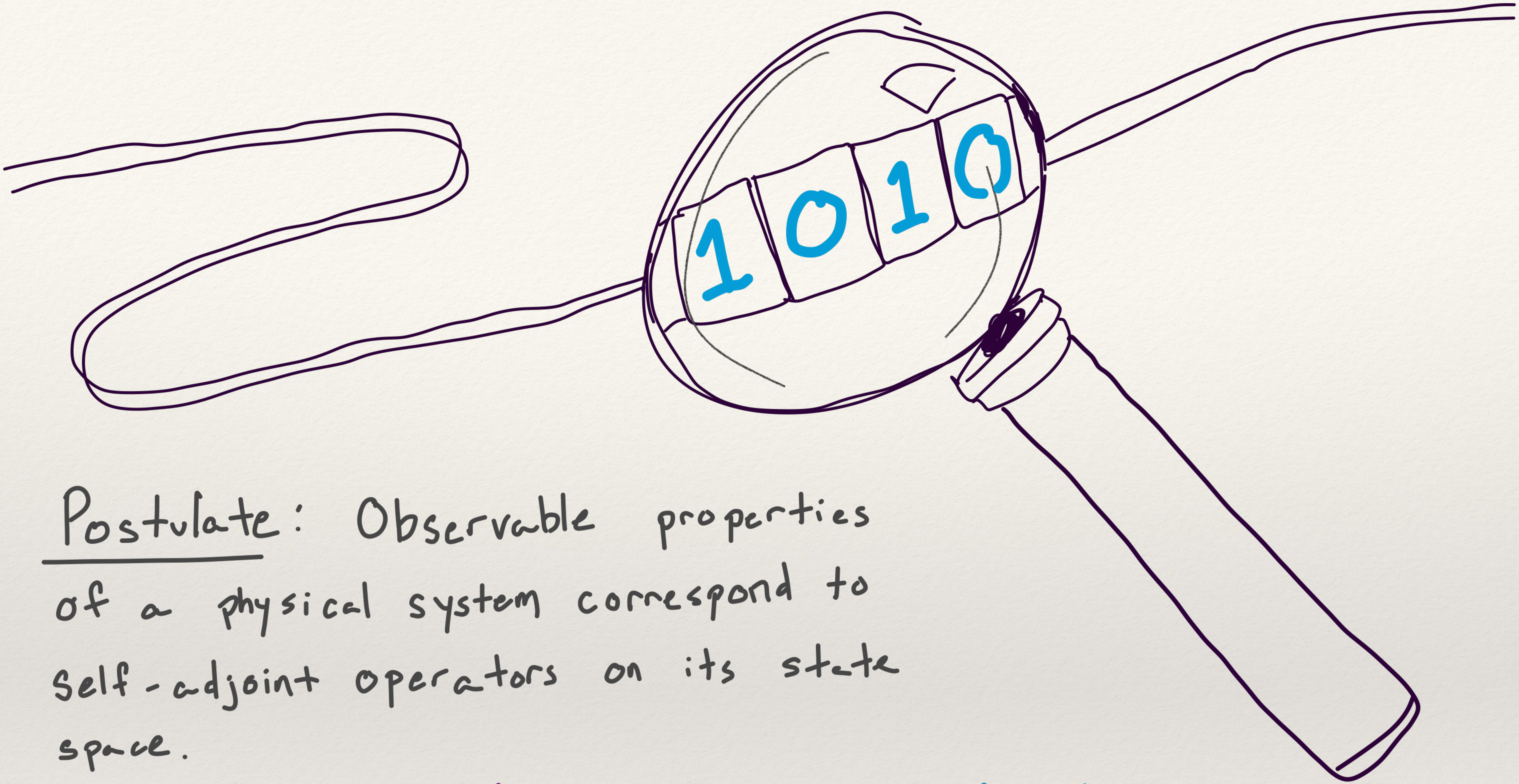
Quantum states are unit vectors in $\mathbb{C}^\Gamma$.

Normalization w.r.t. Hermitian scalar product

$$\langle \cdot | \cdot \rangle : \mathbb{C}^\Gamma \times \mathbb{C}^\Gamma \to \mathbb{C} .$$

States are written as $|\psi\rangle$ where ψ is a label.

e.g. $\{ |x\rangle : x \in \Gamma \}$, $|\text{Primes} < 100\rangle$

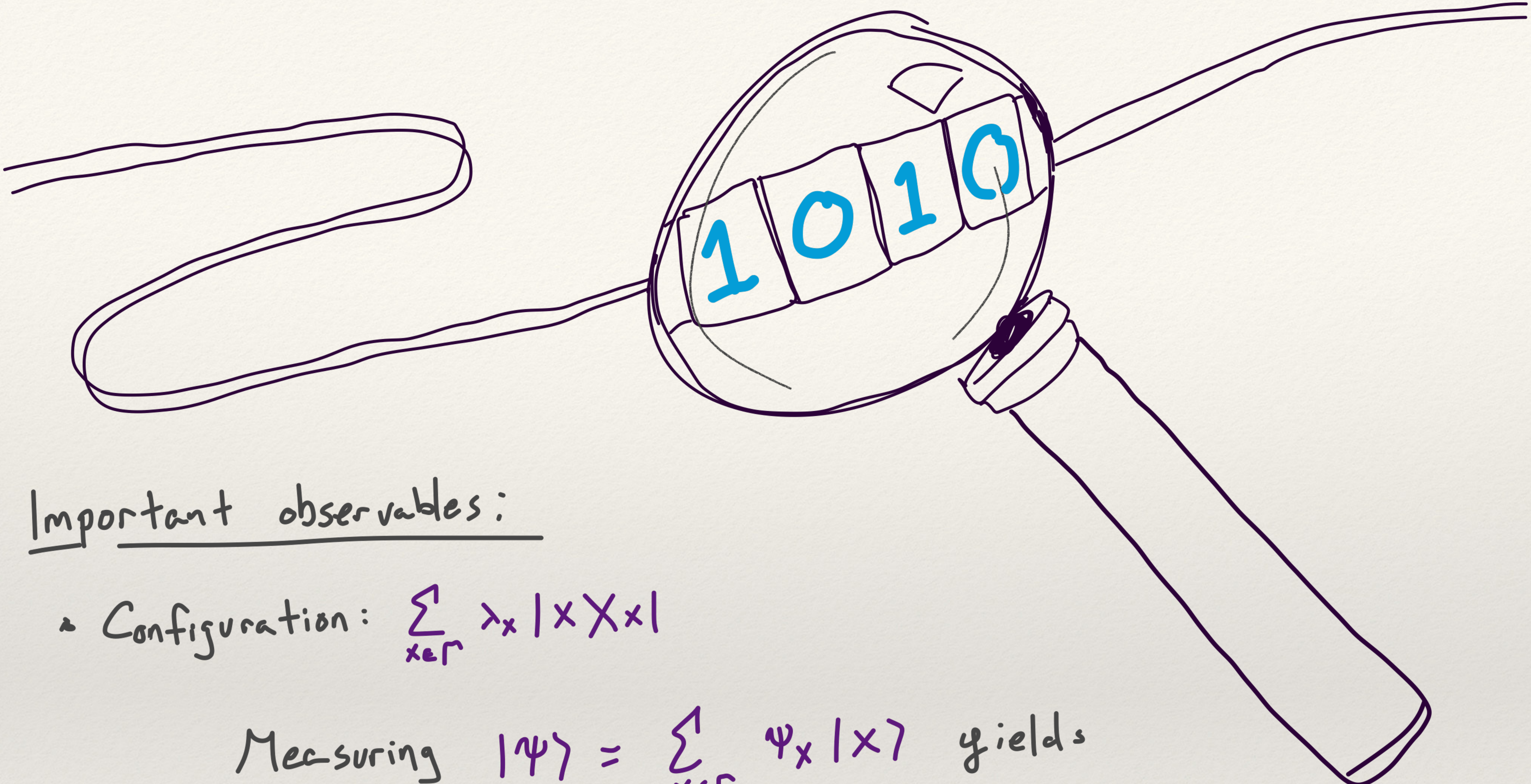


Postulate: Observable properties of a physical system correspond to self-adjoint operators on its state space.

$$\sum_i \lambda_i P_i \leftarrow \text{Projector } (P_i^2 = P_i)$$

$\uparrow$ real eigenvalue

Observe λ_i with probability $\langle \psi | P_i | \psi \rangle$.



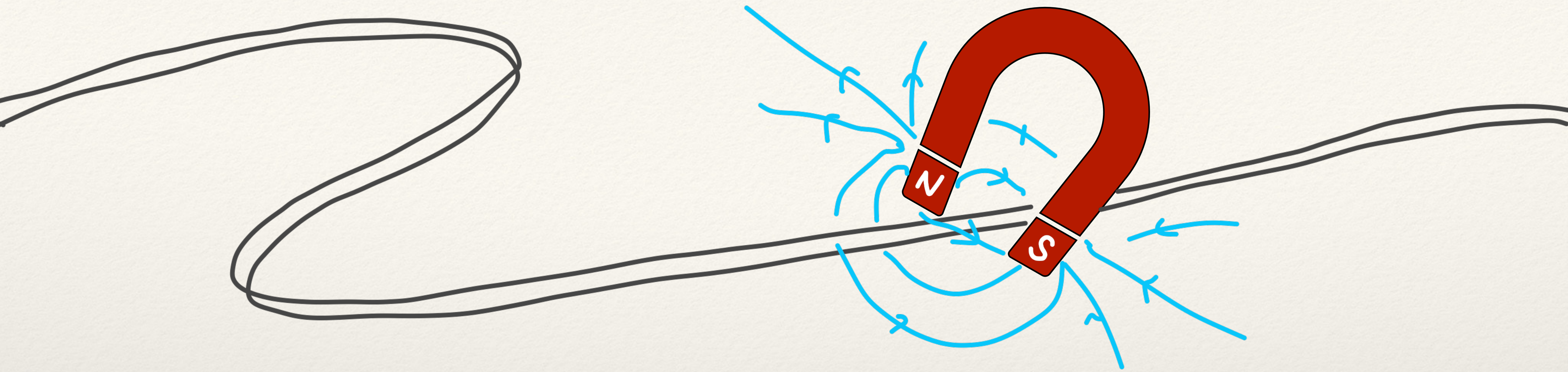
Important observables:

- Configuration: $\sum_{x \in \Gamma} \lambda_x |x\rangle\langle x|$

  Measuring $|\psi\rangle = \sum_{x \in \Gamma} \psi_x |x\rangle$ yields

  λ_x with probability $\langle\psi|x\rangle\langle x|\psi\rangle = |\psi_x|^2$.

- Total energy / Hamiltonian : H



States evolve in time in accordance with the Schrödinger equation:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

the Hamiltonian

In principle, time evolution can generate any unitary transformation.

$$UU^\dagger = U^\dagger U = I$$

Quantum circuits

Sequential composition

$$|\psi\rangle \;-\boxed{U_0}-\boxed{U_1}-\; U_1U_0|\psi\rangle$$

Parallel composition

$$|\psi_0\rangle \;-\boxed{U_0}-\; U_0|\psi_0\rangle$$

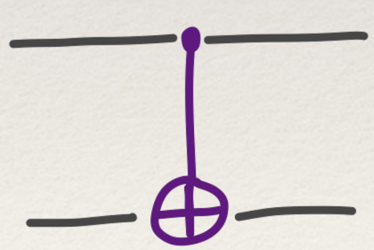
$$|\psi_1\rangle \;-\boxed{U_1}-\; U_1|\psi_1\rangle$$

$$|\psi_0\rangle \otimes |\psi_1\rangle \longmapsto (U_0 \otimes U_1)(|\psi_0\rangle \otimes |\psi_1\rangle)$$

Universal gate set

$|0\rangle$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$-\boxed{H}-$$

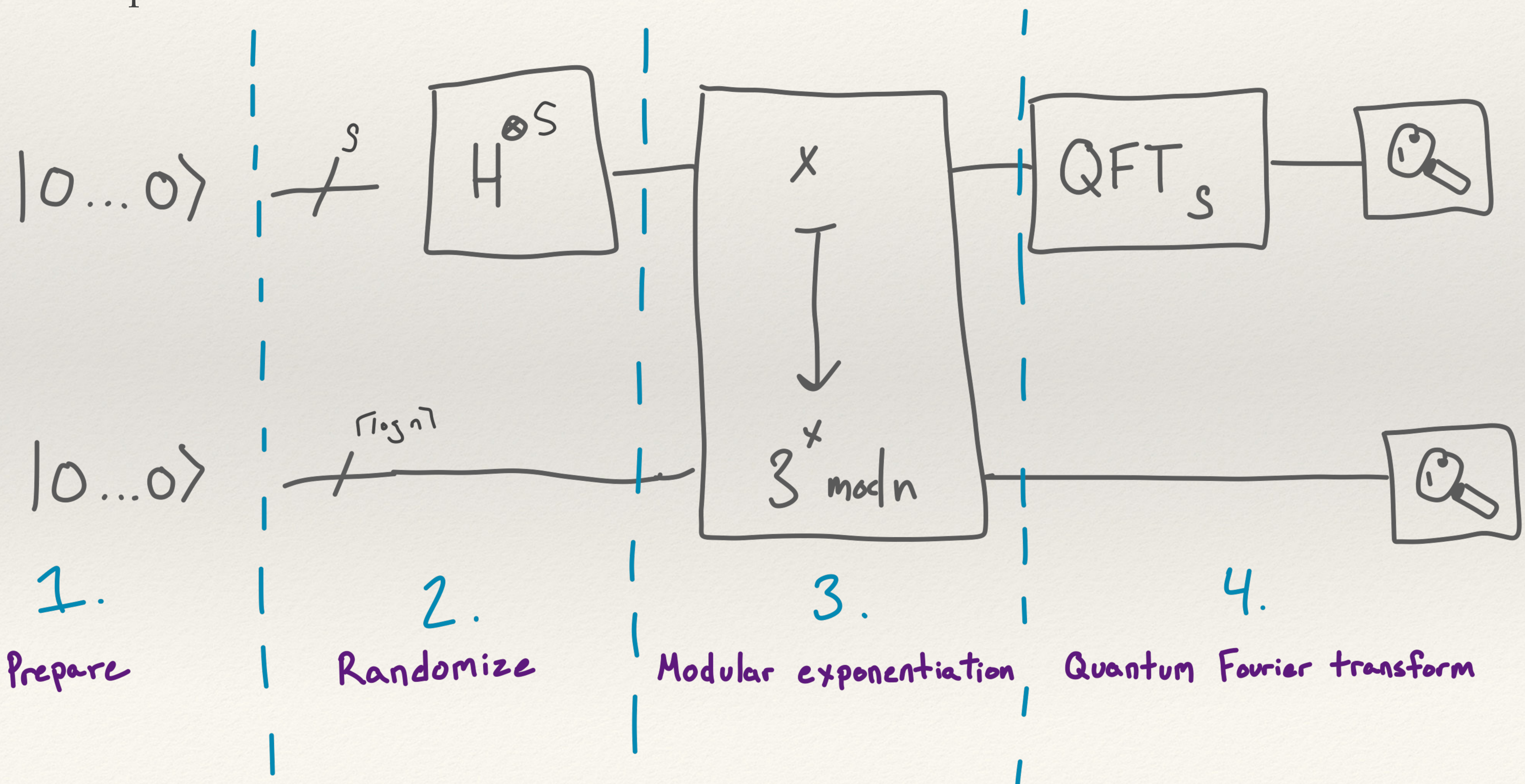
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

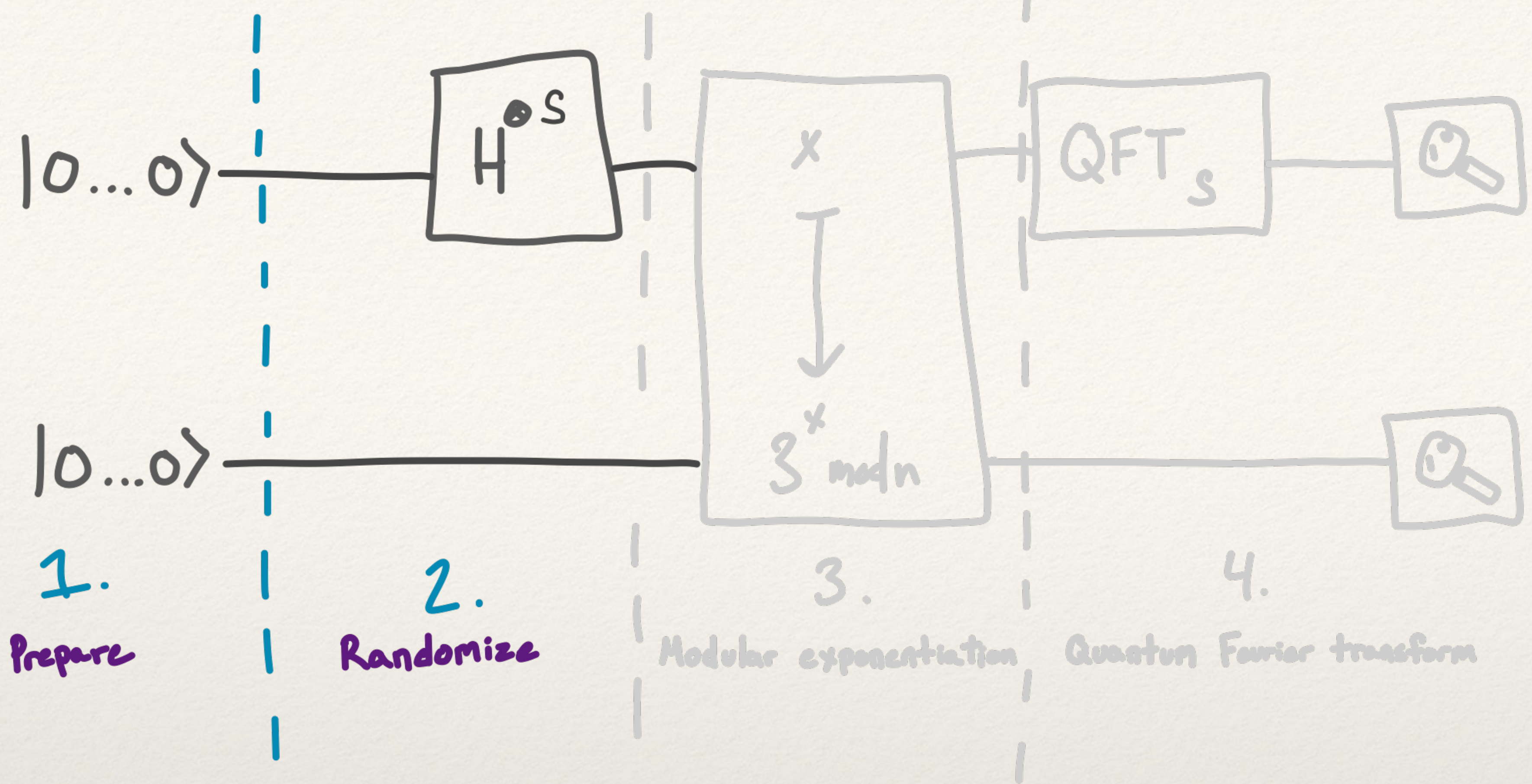
$$-\boxed{T}-$$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Circuit for Shor's algorithm

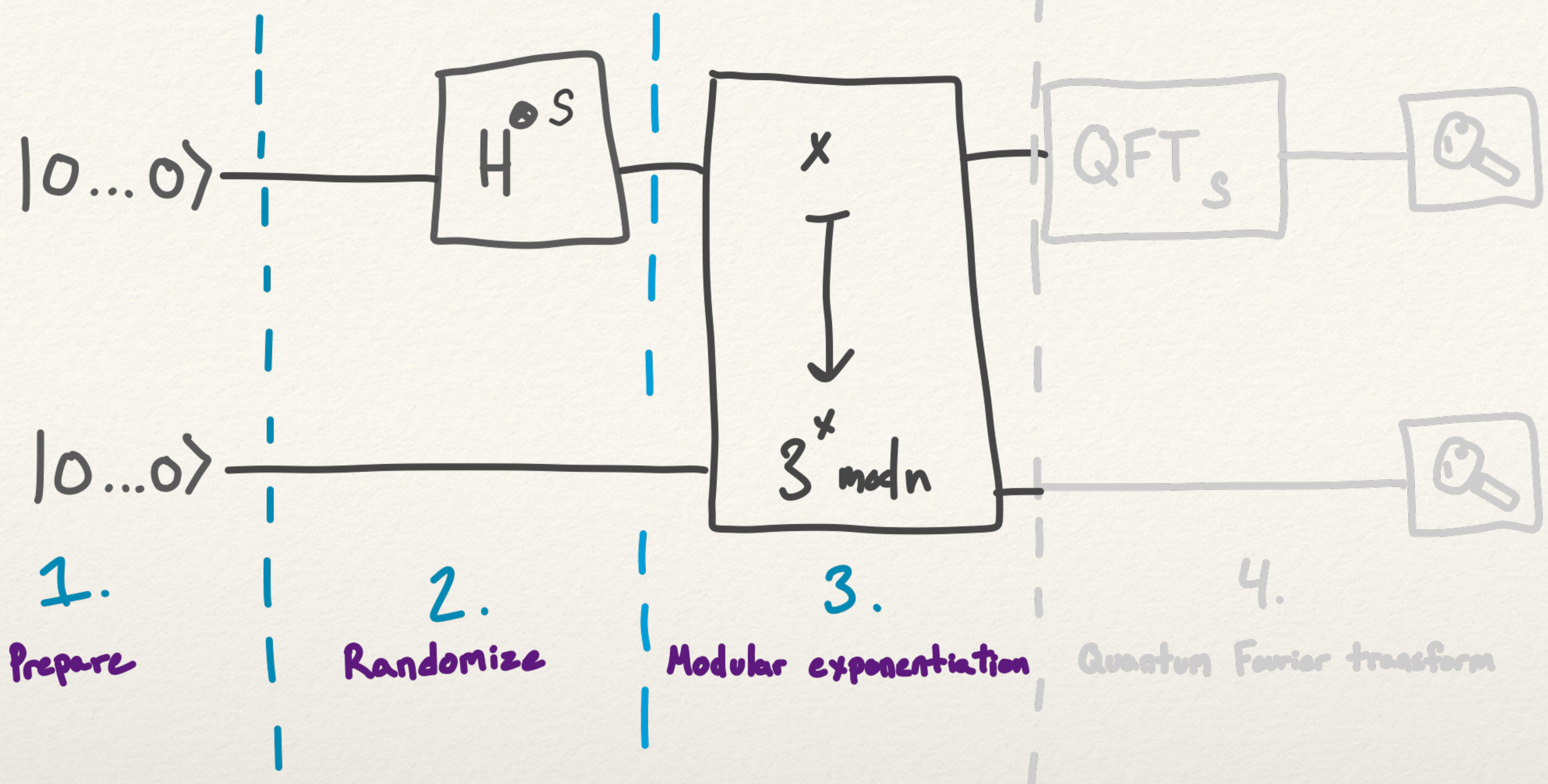
Here n is the number that we want to split and $s = 2\lceil \log_2(n) \rceil$.
This particular circuit finds the order of 3.



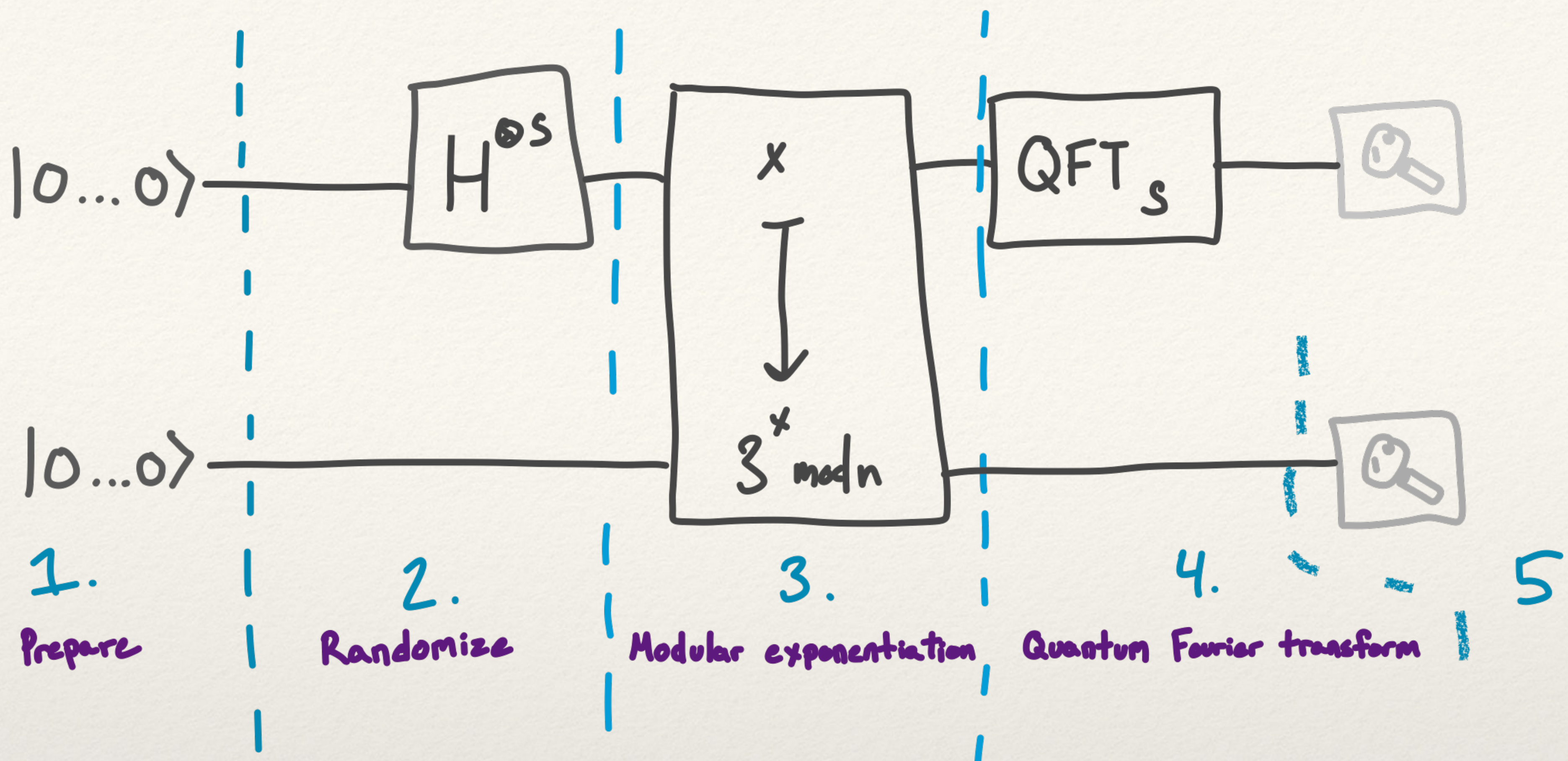


$$|\psi_2\rangle = \left(\frac{1}{\sqrt{2^s}}\sum_{x=0}^{2^s-1}|x\rangle\right)\otimes|0\ldots0\rangle$$

Uniform superposition
over s bit strings



$$|\psi_3\rangle = \frac{1}{\sqrt{2^s}} \sum_x |x\rangle \otimes |3^x \bmod n\rangle$$



$$|\Psi_3\rangle = \frac{1}{\sqrt{2^s}} \sum_x |x\rangle \otimes |3^x \bmod n\rangle$$

$$|x\rangle \xmapsto{QFT_s} \frac{1}{\sqrt{2^s}} \sum_{y=0}^{2^s-1} \omega^{xy} |y\rangle$$

bitstring

Primitive 2^s-th root of unity

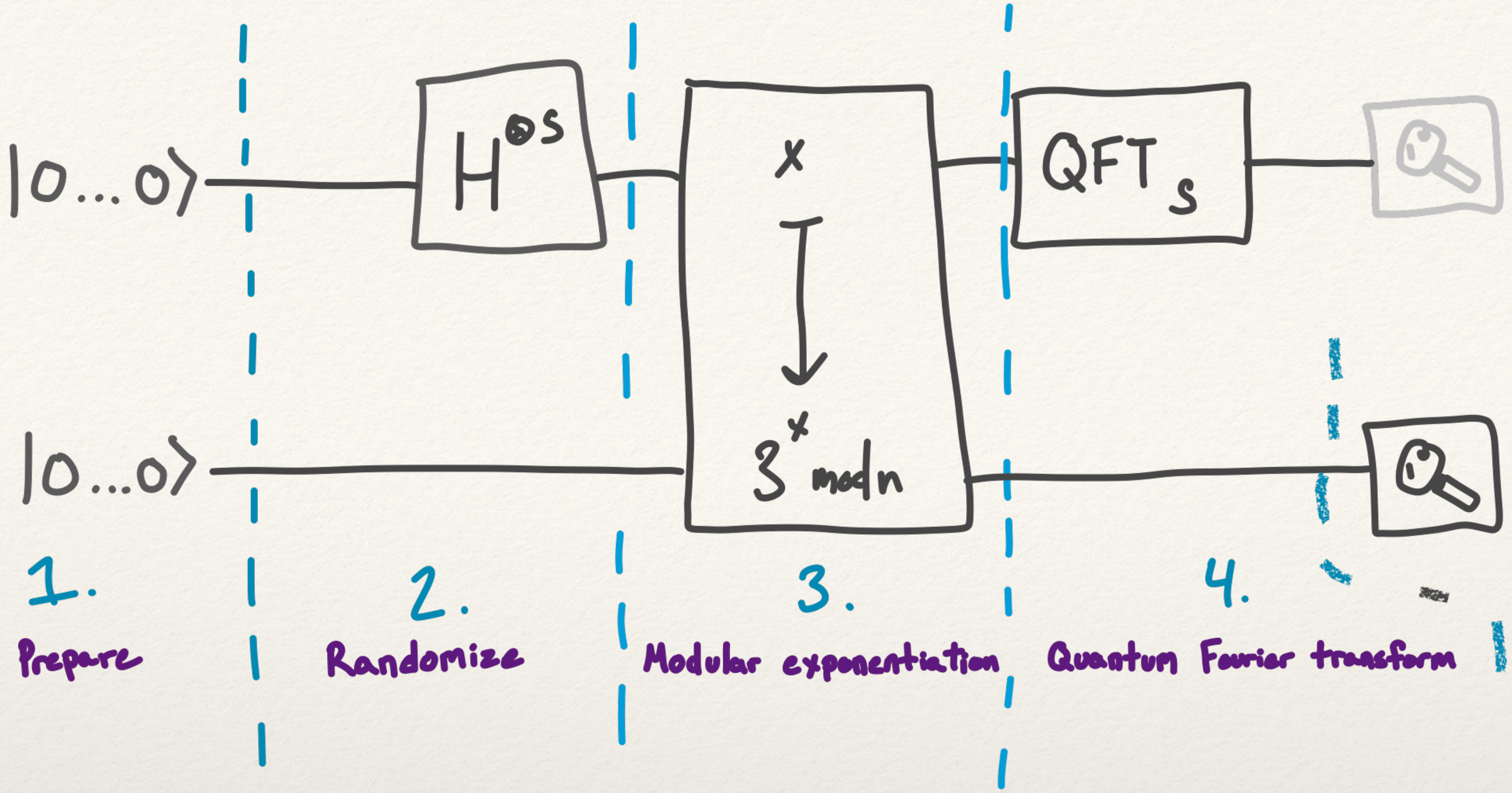
$\omega = \exp(2\pi i / 2^s)$

$$|\Psi_4\rangle = \frac{1}{2^s} \sum_{x=0}^{2^s-1} \sum_{y=0}^{2^s-1} \omega^{xy} |y\rangle \otimes |3^x \bmod n\rangle$$

$$= \frac{1}{2^s} \sum_{a \in (\mathbb{Z}/n)^\times} \sum_{x \in \mathcal{X}(a)} \sum_y \omega^{xy} |y\rangle |a\rangle$$

where

$$\mathcal{X}(a) = \{x : 0 \le x < 2^s \wedge 3^x \equiv a \pmod n\}.$$



$$\mathcal{X}(a) = \{x : 0 \le x < 2^s \wedge 3^x \equiv a \pmod{n}\}.$$

$$= \left\{x_a + kr : 0 \le k \le \left\lfloor \frac{2^s - x_a - 1}{r} \right\rfloor \right\}$$

order of 3

least x s.t. $3^x \equiv a \pmod{n}$

$$|\Psi_4\rangle = \frac{1}{2^s} \sum_a \sum_{x \in \mathcal{X}(a)} \sum_y \omega^{xy} |y\rangle \otimes |a\rangle$$

Measurement yields a'.

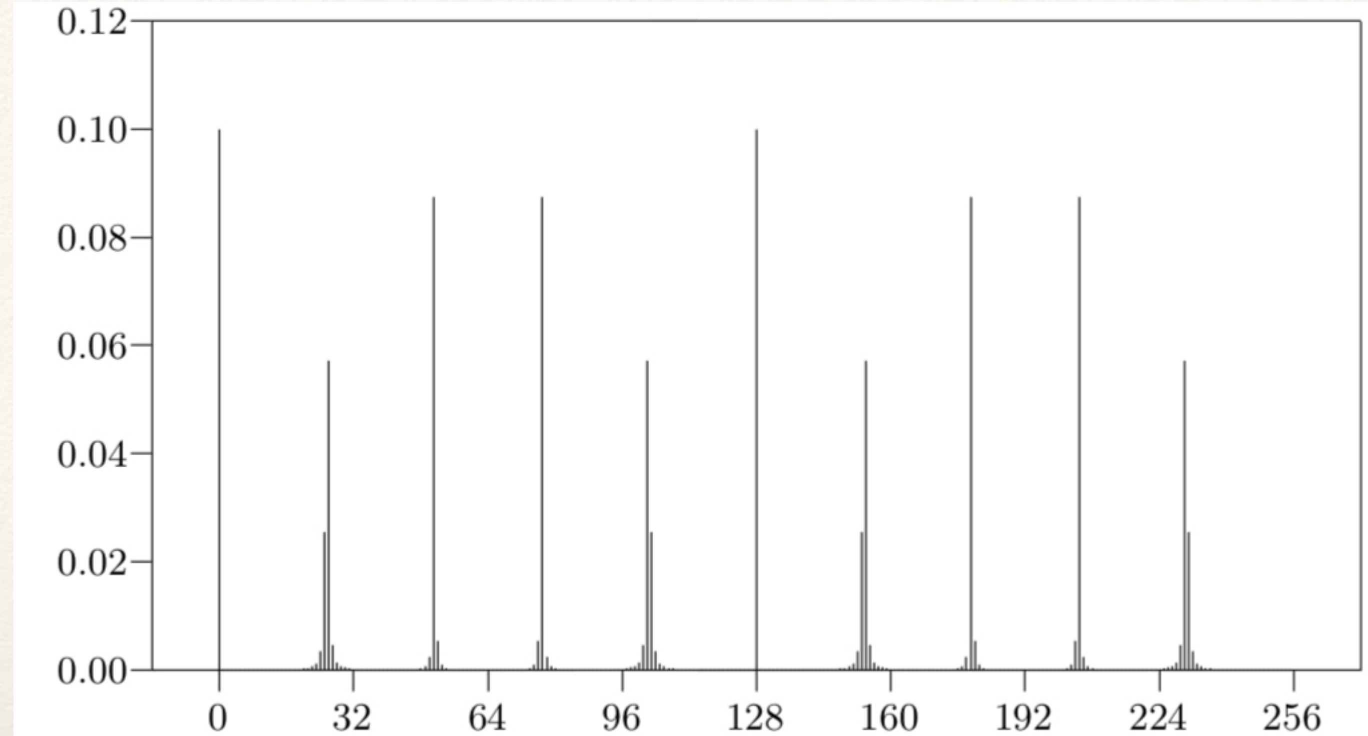
$$|\Psi_5\rangle \propto \sum_{x \in \mathcal{X}(a')} \sum_y \omega^{xy} |y\rangle$$

$$= \omega^{x_a} \sum_y \left(\sum_{k=0}^{\left\lfloor \frac{2^s - x_a - 1}{r} \right\rfloor} (\omega^{ry})^k \right) |y\rangle$$

$$\Pr[y] \propto \left| \sum_{k=0}^{L} (\omega^{ry})^k \right|^2$$

$$L = \left\lfloor \frac{2^s - x_0 - 1}{r} \right\rfloor$$

$\Pr[y]$



y

Shor shows that we are likely to obtain a y for which there exists d such that

$$\left| \frac{y}{2^s} - \frac{d}{r} \right| \leq \frac{1}{2^{s+1}}.$$

Theorem : Let $\alpha \in \mathbb{R}$, and let $\frac{a}{b} \in \mathbb{Q}$ with a and b coprime. If

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{b^2},$$

then $\frac{a}{b}$ appears as a convergent in the continued fraction expansion of α.

Hence the choice of s with $2^s \approx n^2$.

Gate cost of Shor's algorithm

- Gate count is dominated by modular exponentiation.
- If M(t) is the cost of t-bit multiplication then total gate count is $$O(sM(\log n))\,.$$
- Assuming fast multiplication and using $s = 2\lceil \log n \rceil$ as Shor recommends, this is $O(\log^{2+\epsilon} n)$.

Post-quantum RSA

Bernstein—Henninger—Lou—Valenta recently proposed a variant of RSA with:

- $n = p_1 p_2 \cdots p_k$,
- primes of bit length $(\log \log n)^{2+\epsilon}$
- and operations (key generation, encryption, and decryption) that all cost

$$(\log n)(\log \log n)^{O(1)}.$$

bit operations.

Post-quantum RSA

Example

$$n = p_1 p_2 \cdots p_{2^{31}}$$

p_i is a 4096 bit prime for all i

Efficiency:

> *"Our heterogeneous cluster was able to generate primes at a rate of 750–1585 primes per core-hour. Generating all 2^31 primes took approximately 1,975,000 core-hours. In calendar time, prime generation completed in four months running on spare compute capacity of a 1,400-core cluster."*

Security:

> *"Each multiplication modulo n inside Shor's algorithm then uses 2^56 qubit operations, and overall Shor's algorithm consumes an astonishing 2^100 qubit operations."*

Can we reduce ‘s’ in Shor’s algorithm?

- pqRSA assumes M(2^43) = 2^56, and that the modular exponentiation step Shor’s algorithm requires s = 2^44 multiplications. Only clear path to an improved attack is to reduce s.
- One strategy: replace 3 with an element of smaller expected order and take s to be the square of the expected order.
- I don’t see how to do it.

Multi-power post-quantum RSA

An easier problem?

- $n = p_1^{\pi_1} p_2^{\pi_2} \cdots p_k^{\pi_k}$,
- primes of bit length $(\log \log n)^{2+\epsilon}$
- and operations (key generation, encryption, and decryption) all still cost

$$(\log n)(\log \log n)^{O(1)}.$$

bit operations (but fewer primes are needed).

Multi-power pqRSA

- Consider the order of $3^n \bmod n$

$$\varphi(n) = \varphi\left(p_1^{\pi_1} p_2^{\pi_2} \cdots p_k^{\pi_k}\right) = \prod p_i^{\pi_i - 1}(p_i - 1)$$

$$3^n = 3^{\prod p_i^{\pi_i}} \equiv \left(3^{\prod p_i^{\pi_i - 1}}\right)^{\prod_i p_i} \pmod{n}$$

Has order dividing

$$\prod p_i - 1 \approx \exp(k(\log \log n)^{2+\epsilon})$$

Multi-power pqRSA

Example (also with one terabyte n)

$$n = p_1^2 p_2^3 p_3^5 p_4^7 \cdots p_{20044}^{225287}$$

p_i is a 4096 bit prime for all i

In Shor's algorithm we can take $s = 2 \cdot 4096 \cdot 20044 \approx 2^{27}$

Again assume M(2^43) = 2^56, then Shor's algorithm costs 2^83 qubit operations.

But wait! The precomputation, computing 3^n mod n, costs 2^99 bit operations!

What is the most expensive part of this attack?

NIST post-quantum cryptography standardization effort

- Large effort to choose new cryptography (public key encryption, digital signatures, and key encapsulation mechanisms) to replace RSA and other systems vulnerable to quantum attack.
- Over 60 proposals, many sacrificing size/efficiency for security against quantum attacks.
- Better cost analysis of Shor's algorithm will help us tune these other systems.

Thanks!

Quantum computing

- 1961: Landauer considers necessity of energy dissipation in computing processes. Physically irreversible operations necessarily dissipative.
- 1973: Bennett constructs logically reversible Turing machine. Does not propose a physically reversible process.
- 1982: Benioff gives a (non-dissipative) quantum Hamiltonian description of a Turing machine. Some physically unrealistic features.
- 1985: Feynman gives a (non-dissipative) quantum Hamiltonian description of an arbitrary reversible circuit.
- 1984: Zurek shows that Benioff and Feynman's proposals can be made reliable using dissipative error correction.
- 1989: Deutsch proposes (dissipative) circuits with gates that generate the full unitary group.
- 1993: Yao introduces "gate count" as a measure of quantum circuit complexity