Quantum Cryptanalysis in the RAM Model: Claw-Finding Attacks on SIKE

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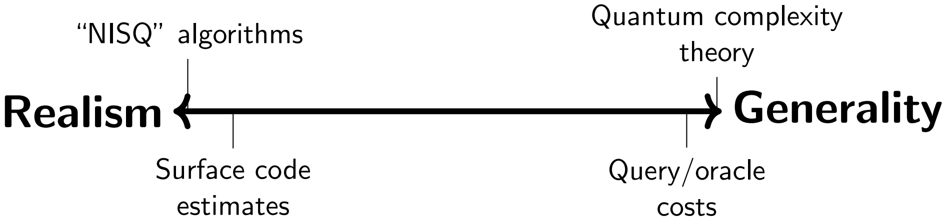
Models of quantum computers

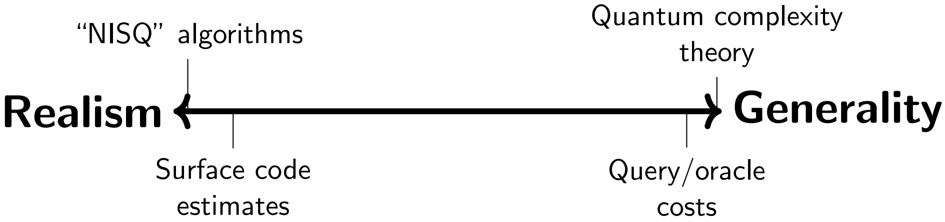
- NIST is working on post-quantum public key standards
- This requires **quantum cryptanalysis**
- This requires models of quantum computers

How do you imagine a quantum computer?

Quantum cost analysis

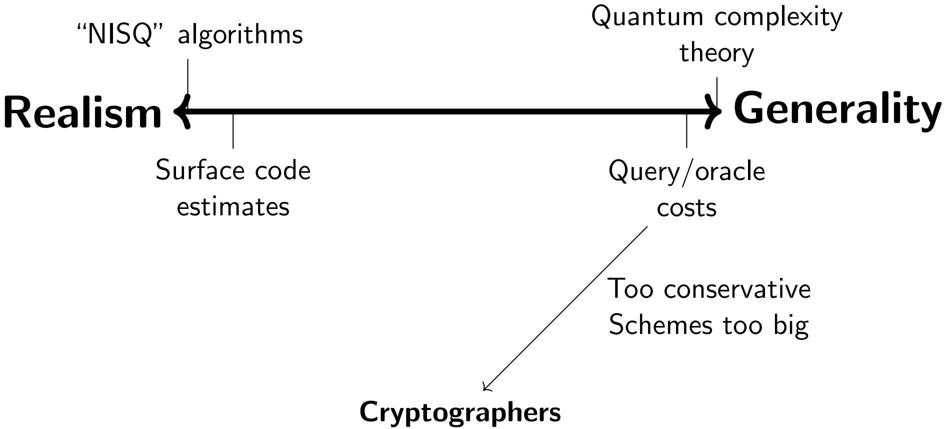


Quantum cost analysis



Cryptographers

Quantum cost analysis



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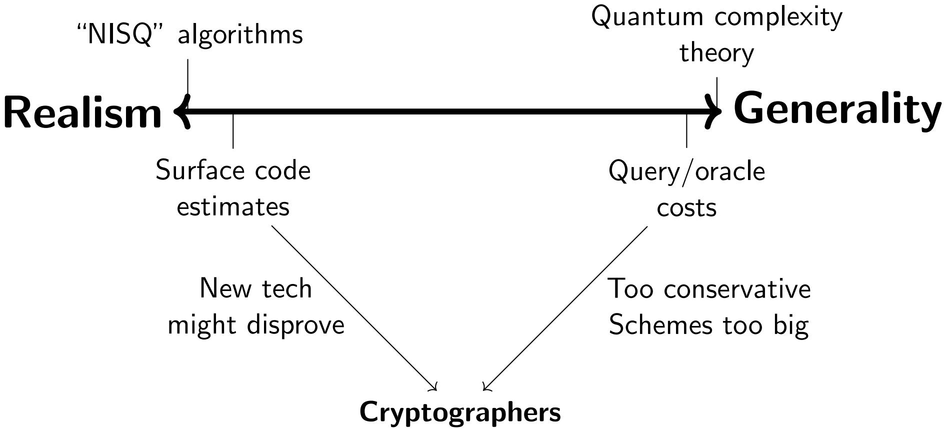

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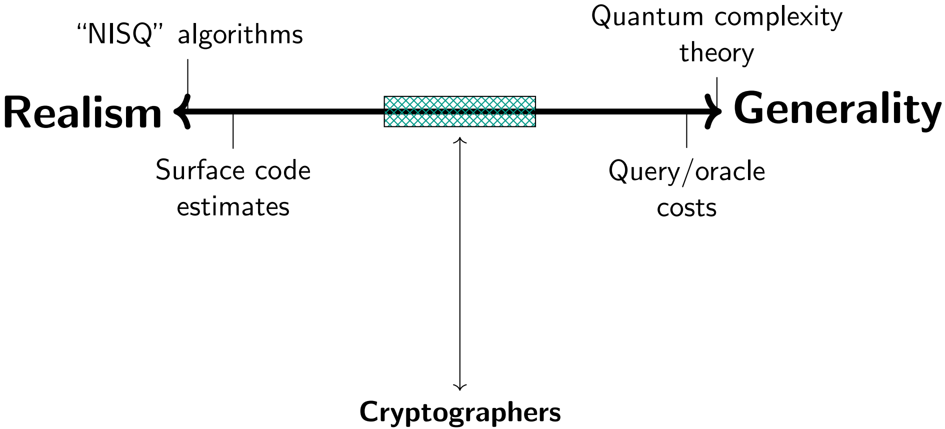

Outline

Goal 1: Fairly compare classical and quantum resources

How do we compare a quantum bit of security to a classical bit of security?
How do we cost mixed classical/quantum algorithms like Kuperberg's sieve?

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How do we compare a quantum bit of security to a classical bit of security?
How do we cost mixed classical/quantum algorithms like Kuperberg's sieve?

Previous work: Analysis of Brassard-Høyer-Tapp (BHT)

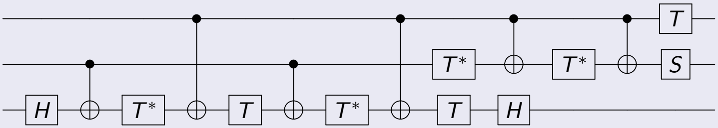
- BHT provided a quantum collision-finding algorithm with quantum access to classical memory.
- Bernstein argued van Oorschot-Wiener is more efficient after fully accounting for memory costs.

Brassard, Høyer, Tapp. 1997. Quantum Algorithm for the Collision Problem

Bernstein. 2009. Cost analysis of hash collisions: Will quantum computers make SHARCS obsolete?

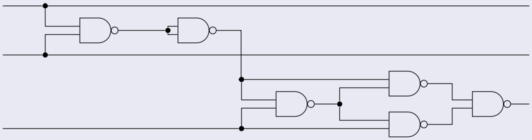
Goal 2: View gates as processes

Quantum



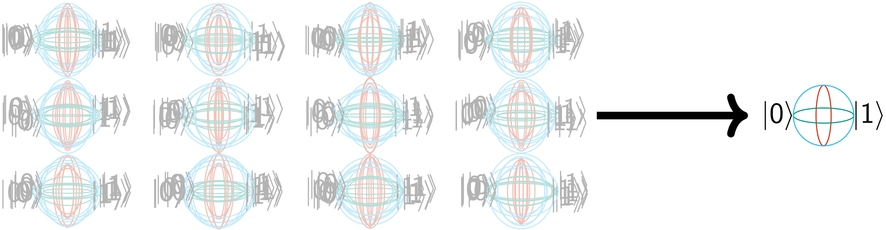
Time→

Classical



Time/Space→

Goal 3: Include error correction



Many physical qubits

One logical qubit

Memory peripheral framework

Main Idea

Model computation as “memory” acted on by a “memory controller”.

Examples:

- Turing machine: head + tape
- RAM: CPU + random access memory
- Quantum circuit: Random access machine + qubits

Premises:

1. **Memory** is a physical system that changes over time
2. A **memory controller** interacts with a memory
3. The **cost** of a computation is the number of interactions

Premise 1: Memory is a physical system

Free evolution

Caused by:

- Noise
- Ballistic computation

Costly evolution

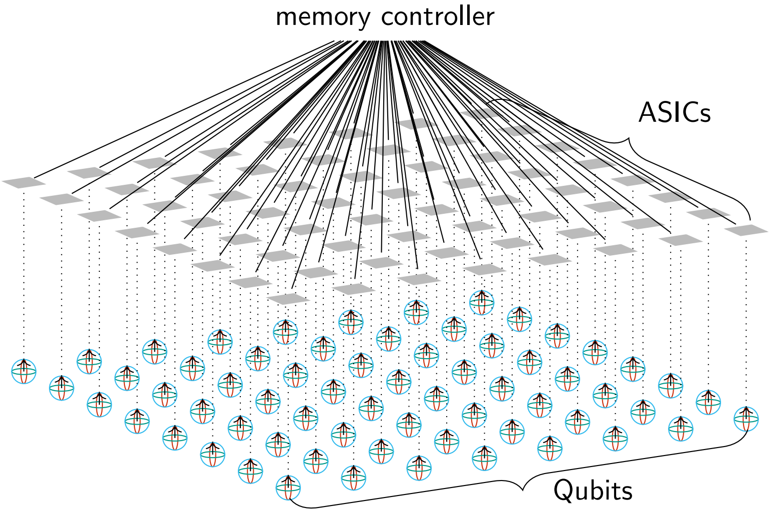
Caused by the controller.

We model a quantum computer as a **parallel random access machine** with new instructions for quantum gates

- e.g.: apply gate x to qubit y at time t

Result: quantum algorithms are classical programs

Premise 2: Memory controller



Premise 3: Cost

The **cost** of a computation is the number of interactions.

- We ignore the construction cost
- We focus on the cost to the controller

There are opportunity costs: What else could the controller do?

Cost models

We provide physical justifications for two cost models: **G-cost** and **DW-cost**.

Both are qubit memories with a standard universal gate set (Clifford + T).

Differences:

- G-cost: **Passive** error correction.
- DW-cost: **Active** error correction.

Error correction

Passive/Non-volatile memory

To preserve: keep cool.

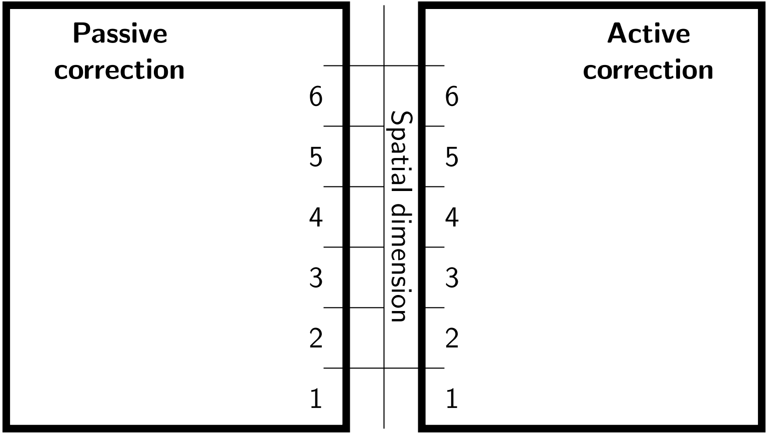
- Paper
- Magnetic discs

Active/Volatile memory

To preserve: continuously refresh.

- DRAM
- Surface codes (quantum)

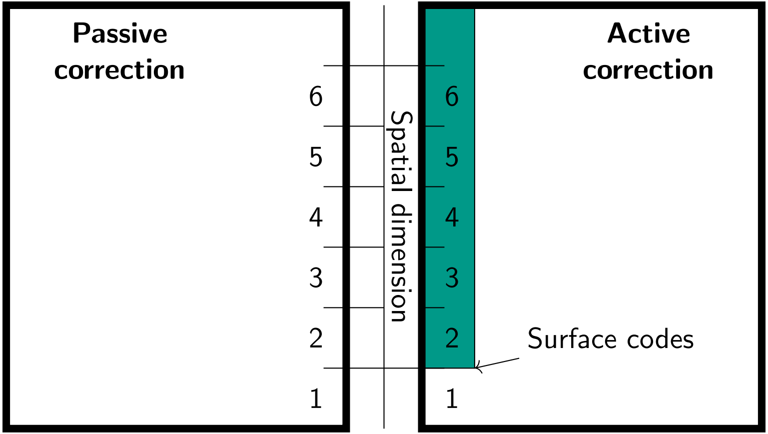
Quantum error correction theory



[1] Bravyi and Terhal. 2009. A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes.
[2] Kitaev. 2003. Fault-tolerant quantum computation by anyons.

Dennis, Kitaev, Landahl, Preskill. 2002. Topological Quantum Memory.

Quantum error correction theory



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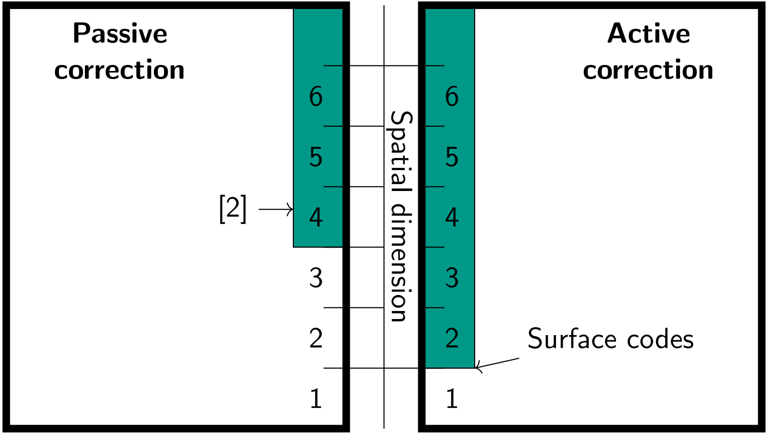

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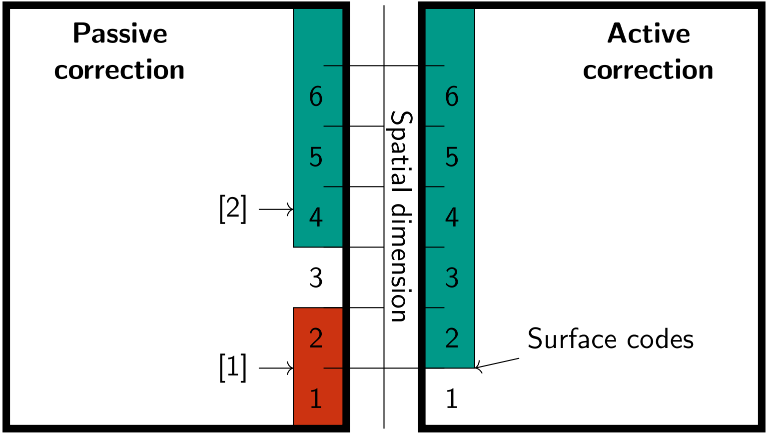

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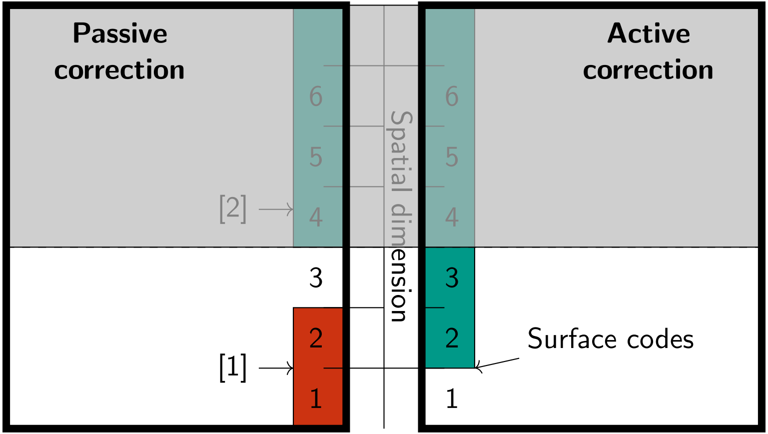

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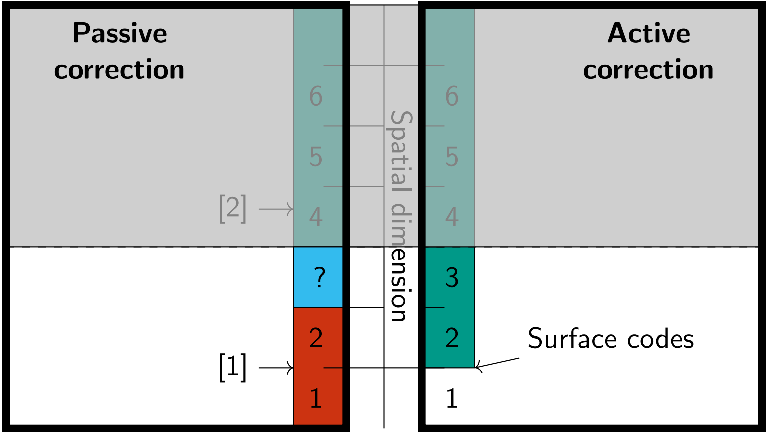

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Costs

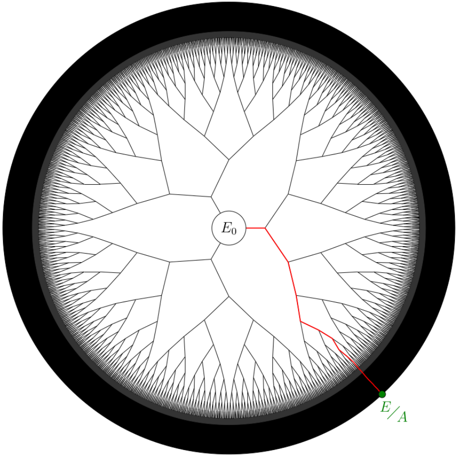
G-cost

- Assumption: **Passive** error correction.
  (Physical, not just technological, assumption)
- Cost: 1 RAM operation per gate
- Total cost: Number of gates ("G")

DW-cost

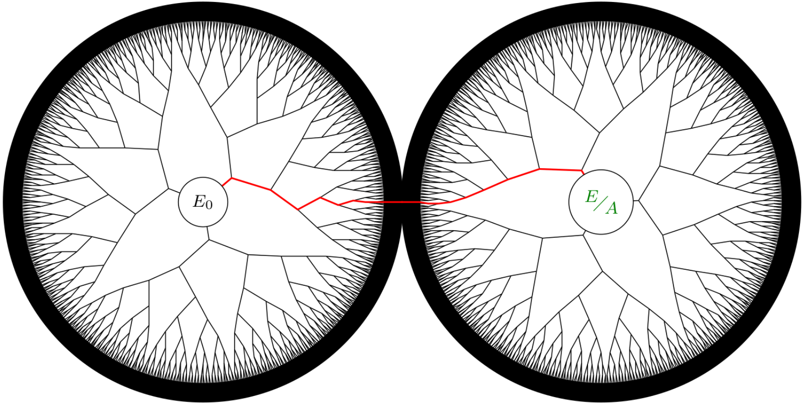
- Assumption: **Active** error correction.
- Cost: 1 RAM operation per qubit per time step
- Total cost: Depth$\times$Width ("DW")

Analysis of SIKE



- E_0 is public parameter, E/A is public key
- Parameterized by a large prime p (e.g. $p \approx 2^{434}$)
- Red path is secret key (length $\log p/2$)

Meet-in-the-middle



Tani's collision-finding algorithm

To find a collision between two functions $f : X \to S$ and $g : Y \to S$:

- Random walk on two Johnson graphs: one over X, the other over Y
- Check for collisions at each step
- Make it quantum!

Johnson graph over X

Vertices: R-element subsets of a fixed set X.
Vertices u and v are adjacent iff $|u \cap v| = R - 1$.

Tani. 2007. An improved claw finding algorithm using quantum walk.

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Query-optimal parameters:

$$R = \# \text{ queries } = \text{ time}$$

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Johnson graph over X

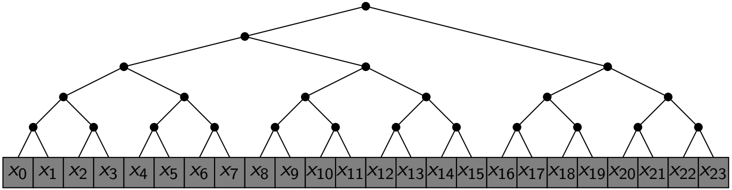
Vertices: R-element subsets of a fixed set X.
Vertices u and v are adjacent iff $|u \cap v| = R - 1$.

Query-optimal parameters to attack SIKE:

$$R = \# \text{ queries } = \text{ time } = p^{1/6+o(1)}$$

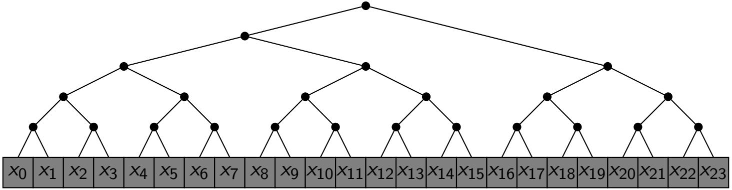
Tani. 2007. An improved claw finding algorithm using quantum walk.

Memory access



Memory access

Classical Query: 9



Memory access

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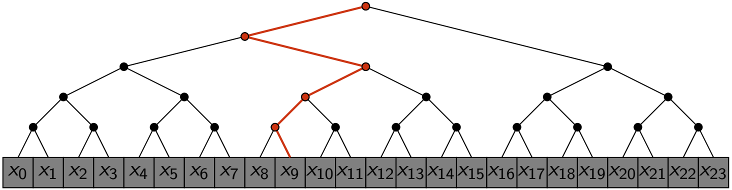
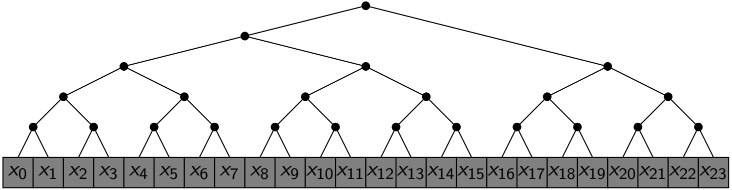


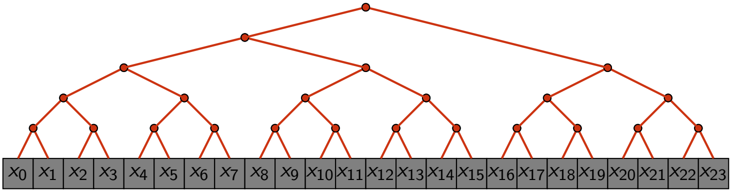
Memory access

Quantum Query:



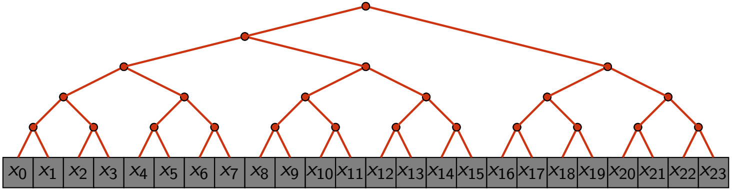
Memory access

Quantum Query: [illegible]



Memory access

Quantum Query:



Analogy for Cryptographers

- Any physical “side channel” leaks information
- Any leaked information decoheres (destroys) the state
- Controller must implement circuits for all possible inputs

Memory costs

For N bits of random-access quantum memory:

Idle memory

- G-cost: Free
- DW-cost: $O(N)$ RAM ops per time step

Random access

- G-cost: $O(N)$ RAM ops
- DW-cost: $O(N \log N)$ RAM ops

Johnson vertex data structure

History independence

For quantum interference between random walk paths, the representation of a vertex must be independent of the path taken.

Bernstein, Jeffery, Lange, Meurer. 2013. Quantum algorithms for the subset-sum problem

Johnson vertex data structure

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For quantum interference between random walk paths, the representation of a vertex must be independent of the path taken.

History-dependent:

- Binary tree as linked list

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Johnson vertex data structure

History independence

For quantum interference between random walk paths, the representation of a vertex must be independent of the path taken.

History-dependent:

- Binary tree as linked list

History-independent:

- Quantum radix tree: superposition over all layouts
- Sorted array: physically in order

Bernstein, Jeffery, Lange, Meurer. 2013. Quantum algorithms for the subset-sum problem

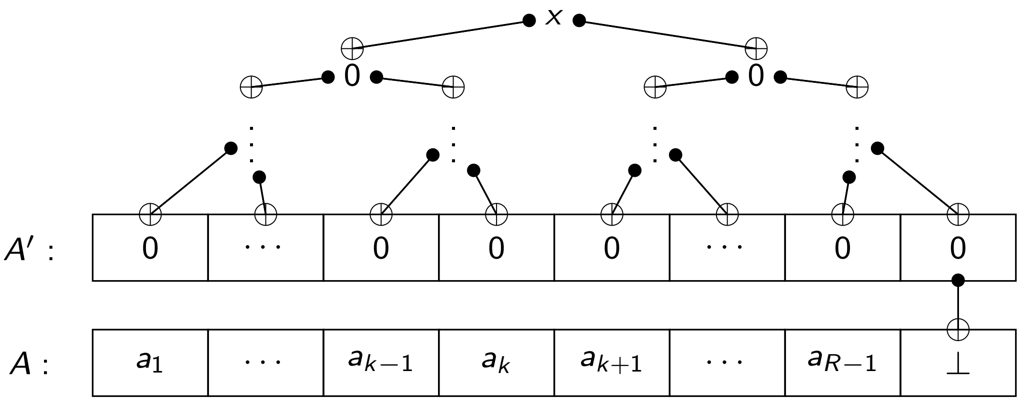
Johnson vertex insertion

Idea: We already pay $O(N)$ for memory access, so pay $O(N)$ to physically sort array:

A' :	0	$\cdots$	0	0	0	$\cdots$	0	0
A :	a_1	$\cdots$	a_{k-1}	a_k	a_{k+1}	$\cdots$	a_{R-1}	$\perp$

Johnson vertex insertion

1. "Fan out" an input x



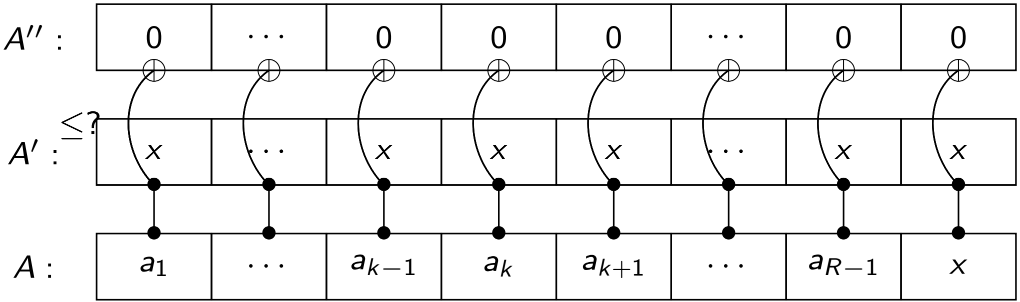
Johnson vertex insertion

1. "Fan out" an input x

A' :	x	$\cdots$	x	x	x	$\cdots$	x	x
A :	a_1	$\cdots$	a_{k-1}	a_k	a_{k+1}	$\cdots$	a_{R-1}	$\perp$

Johnson vertex insertion

2. Compare all elements simultaneously



Johnson vertex insertion

2. Compare all elements simultaneously

A'' :

0	$\cdots$	0	1	1	$\cdots$	1	1

A' :

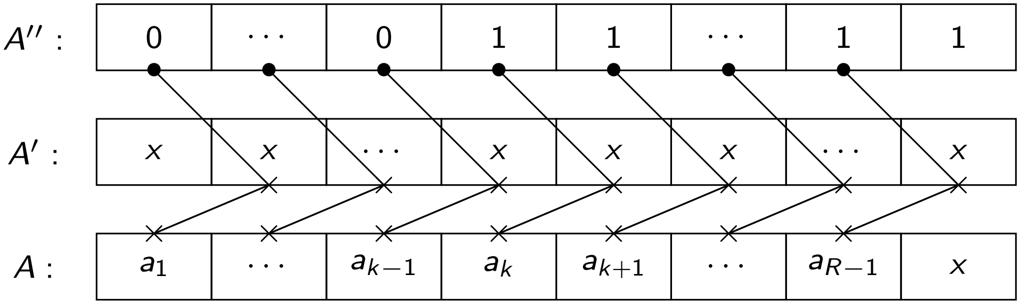
x	$\cdots$	x	x	x	$\cdots$	x	x

A :

a_1	$\cdots$	a_{k-1}	a_k	a_{k+1}	$\cdots$	a_{R-1}	x

Johnson vertex insertion

3. Conditionally swap “up”



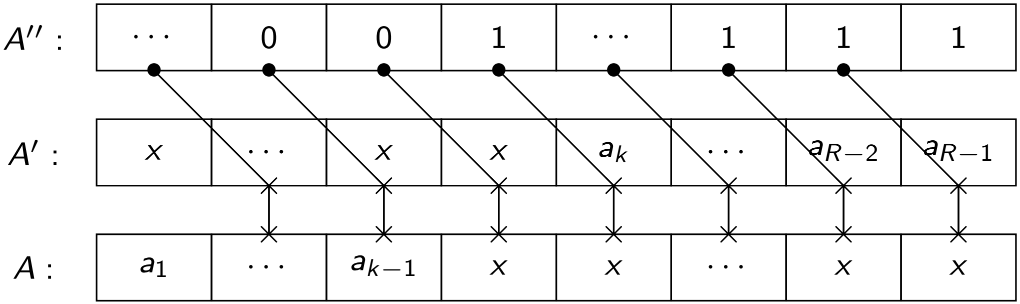
Johnson vertex insertion

3. Conditionally swap "up"

A'' :	0	$\cdots$	0	1	1	$\cdots$	1	1
A' :	x	$\cdots$	x	x	a_k	$\cdots$	a_{R-2}	a_{R-1}
A :	a_1	$\cdots$	a_{k-1}	x	x	$\cdots$	x	x

Johnson vertex insertion

4. Conditionally swap “down”



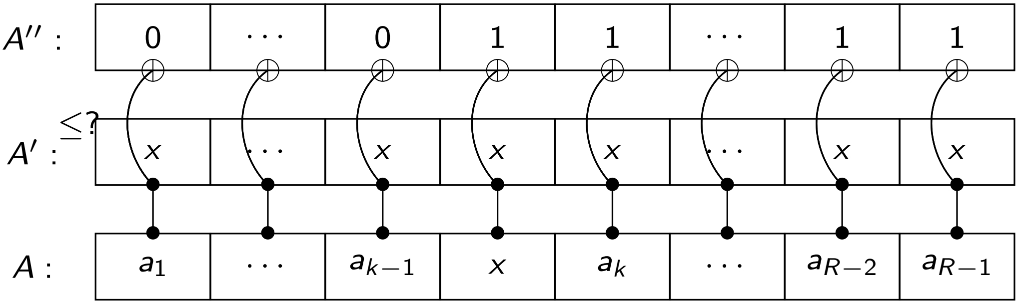
Johnson vertex insertion

4. Conditionally swap "down"

A'' :	$\cdots$	0	0	1	$\cdots$	1	1	1
A' :	x	$\cdots$	x	x	x	$\cdots$	x	x
A :	a_1	$\cdots$	a_{k-1}	x	a_k	$\cdots$	a_{R-2}	a_{R-1}

Johnson vertex insertion

5. Clear comparison bit



Johnson vertex insertion

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A'' :	0	$\cdots$	0	0	0	$\cdots$	0	0
A' :	x	$\cdots$	x	x	x	$\cdots$	x	x
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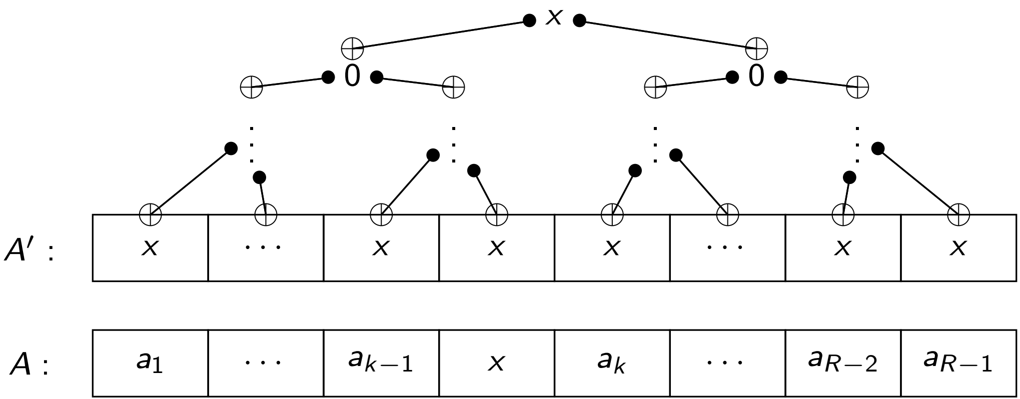
Johnson vertex insertion

7. Clear fan-out



Johnson vertex insertion

8. Insertion complete

A' :	0	$\cdots$	0	0	0	$\cdots$	0	0
A :	a_1	$\cdots$	a_{k-1}	x	a_k	$\cdots$	a_{R-2}	a_{R-1}

Costs of Tani's algorithm for SIKE

Previous analyses focused on the $p^{1/6}$ query cost of Tani's algorithm.

Using the Johnson vertex data structure, we find the SIKE secret at cost:

	Gates	Depth	Width	DW
Tani (query-optimal)	$p^{1/3+o(1)}$	$p^{1/6+o(1)}$	$p^{1/6+o(1)}$	$p^{1/3+o(1)}$

$$2^{434} \leq p \leq 2^{951}$$

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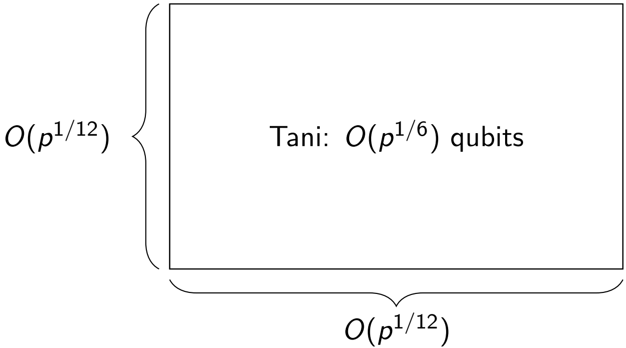
Grover (G-optimal)	$p^{1/4+o(1)}$	$p^{1/4+o(1)}$	$p^{o(1)}$	$p^{1/4+o(1)}$

$$2^{434} \leq p \leq 2^{951}$$

Comparison with parallel Grover

The classical controller can apply gates to every qubit to run Tani's algorithm.
It could instead group them together and run Grover's search algorithm.



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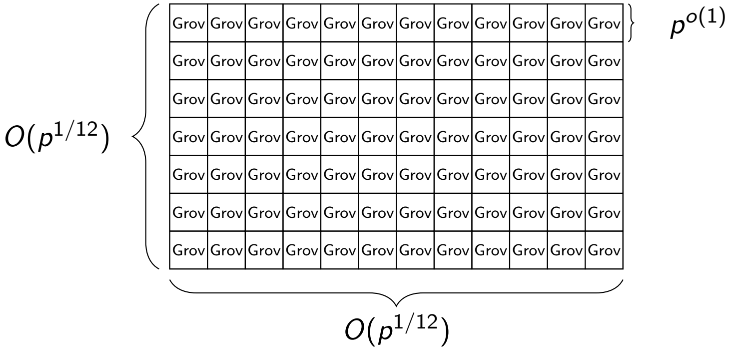
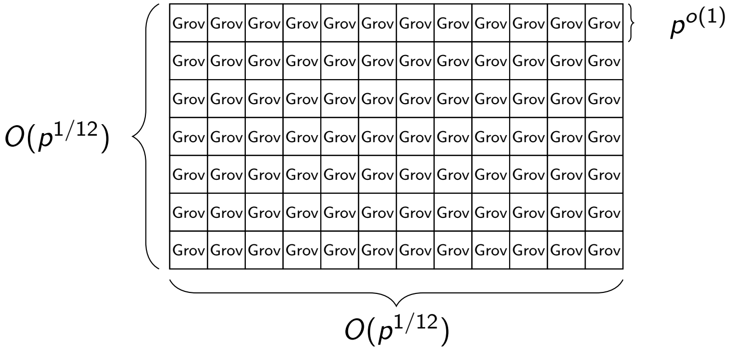


Grover and Rudolph. 2004. How significant are the known collision and element distinctness quantum algorithms

Comparison with parallel Grover

$O(p^{1/6})$ copies of Grover finds isogeny in time $O(p^{1/6})$.



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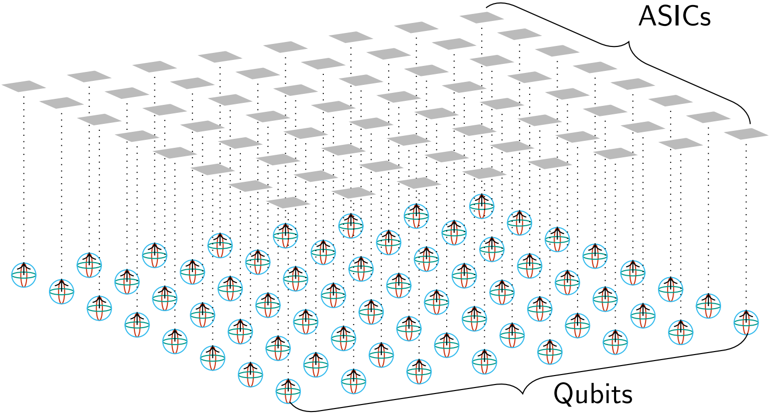
Comparison with van Oorschot-Wiener

- Time/query-optimal Tani has $O(p^{1/6})$ classical control processors.
- We could reprogram these to run van Oorschot-Wiener (VW)



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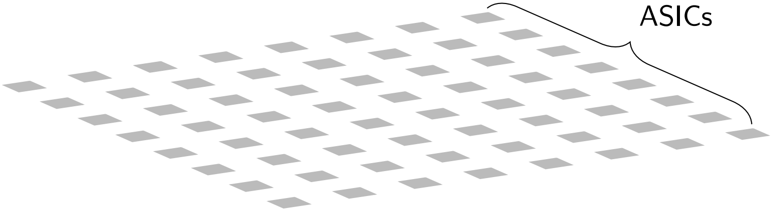

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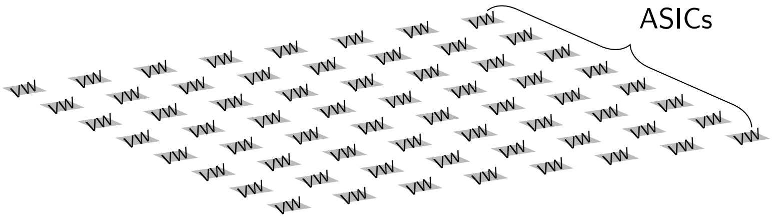

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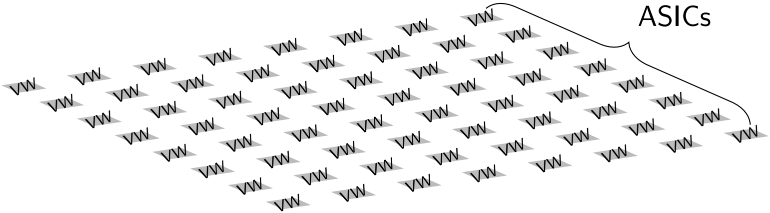
Conclusion

$O(p^{1/6})$ parallel instances of van Oorschot-Wiener find isogeny in time $O(p^{1/8})$, faster than the quantum algorithms.



Memory peripheral framework

1. **Memory** is a physical system that changes over time
2. A **memory controller** interacts with a memory
3. The **cost** of a computation is the number of interactions

Conclusions

- In a quantum computer, qubits are a peripheral of a classical computer.
- Quantum memory access has a linear gate cost.
- Active error correction gives cost to the identity gate.
- SIKE is more secure than previously thought.

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